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Computable Time Concentration of Bandlimited Signals and Systems

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***Abstract*—Turing computability deals with the question of what is theoretically computable on a digital computer, and hence is relevant whenever digital hardware is used. In this paper we study different possibilities to define computable bandlimited signals and systems. We consider a definition that uses finite Shannon sampling series as approximating functions and another that employs computable continuous functions together with an effectively computable time concentration. We discuss the advantages and drawbacks of both definitions and analyze the connections and differences. In particular, we show that both definitions are equivalent for many practically relevant signal classes, e.g. for bandlimited signals with finite energy, but also that there are important differences, such as for the impulse responses of BIBO stable LTI systems.**

***Index Terms*—Bandlimited signal, time concentration, effective approximation, Turing computability, digital and analog signal processing**

I. Introduction

Bandlimited signals play a crucial role in signal processing [2]–[7]. According to Shannon's sampling theorem, a bandlimited signal with finite energy is uniquely determined by its samples taken at the Nyquist rate, and the continuous-time signal can be recovered from the samples by means of the Shannon sampling series [8], [9]. Shannon originally formulated the sampling theorem for bandlimited signals with finite energy. By now, many authors have extended this result in different directions, e.g., to sampling theorems for more general signal spaces [10]–[12], missing samples [13], non-bandlimited signals [14], signals bandlimited in the fractional Fourier transform domain [15]–[17], stochastic processes [18], and multiband signals [19].

In general, sampling theorems are important whenever continuous-time signals have to be converted to discrete-time signals and vice versa. Nowadays, digital technology is ubiquitous and of enormous importance because most signal processing is done in the digital domain.

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Despite the widespread and successful use of digital signal processing it is interesting to study the limitation of digital signal processing, also in light of the recent progress that has been made towards the development of novel computing technologies, such as analog neuromorphic computing and analog quantum computing.

A fundamental issue in the theory of bandlimited signals is the inherent infinite time duration of these signals, which is caused by their band limitation. Often, it is argued that practically relevant bandlimited signals are essentially time limited, in the sense that most of their signal energy is contained in some relevant time interval, or, similarly, that the signal amplitudes are negligible outside of this interval. Thus, the signal concentration in the time domain is of general interest [20]–[22]. The question that we ask is: Can we algorithmically determine the essential time concentration of a bandlimited signal?

In this paper we consider the Bernstein spaces $\mathcal{B}_\pi^p$, i.e., bandlimited signals with finite L^p-norm as characteristic time domain behavior [23]. Often, such signals cannot be represented in closed form, e.g., in optimization tasks or filter design problems [24]. Hence, the approximation of such signals and the control of the approximation error are important.

Nowadays, most signal processing is done on digital hardware, such as microprocessors, field programmable gate arrays (FPGAs), or digital signal processors (DSPs), and hence questions of computability arise. In order to study these questions, we employ the concept of Turing computability. A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules [25], [26]. Although the concept is very simple, a Turing machine is capable of simulating any given algorithm. Turing machines have no limitations in terms of memory or computing time, and hence provide a theoretical model that describes the fundamental limits of any practically realizable digital computer. Computability is a mature topic in computer science [25]–[30]. In the signal processing literature, however, it has not received much attention.

Computability is important for the control of the approximation error if digital hardware is used to compute the signals. One of the key concepts of computability is the effective, i.e., algorithmic control of the approximation error. If a signal is computable, then for every prescribed error tolerance ϵ we can compute an approximation that is ϵ-close to the desired signal. This is illustrated in Fig. 1. In contrast to classical approximation theory, where the mere mathematical existence of an approximation is sufficient, the essential point for computability is that, for any given error tolerance $\epsilon > 0$,

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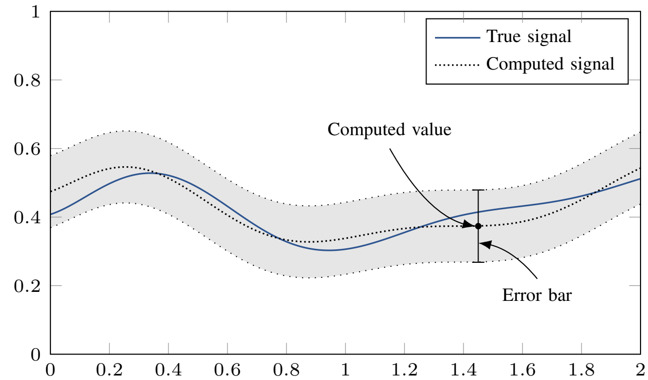

Fig. 1. For a computable signal we can always determine an error bar and then can be sure that the true value lies within the specified error range. Illustration similar to [31].

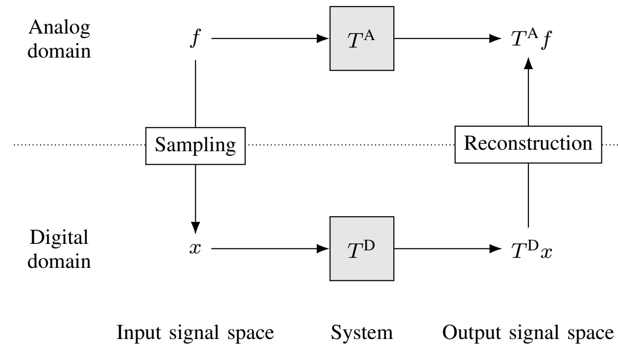

Fig. 2. Processing of analog signals in the digital domain. Illustration similar to [36].

the approximation can be algorithmically computed in a finite number of steps.

In [31]–[35] computability was introduced for bandlimited signals in $\mathcal{B}_\pi^p$, using a natural approach that reflects the special role of the Shannon sampling series. More specifically, according to this definition a signal $f \in \mathcal{B}_\pi^p$ is called computable if

1) there exists an algorithm that computes a sequence $\{f_n\}_{n\in\mathbb{N}}$ of finite Shannon sampling series in $\mathcal{B}_\pi^p$, and
2) the approximation error can be effectively controlled, i.e., we have $\|f - f_n\|_{\mathcal{B}_\pi^p} \leq 2^{-n}$ for all $n \in \mathbb{N}$.

The motivation for using the Shannon sampling series in this definition has been twofold. First, the Shannon sampling series plays a fundamental role in signal processing, especially in the theory of bandlimited signals and, second, the finite Shannon sampling series with rational coefficients can be computed on a digital computer, while having an effective control of the approximation error. The advantages of the above definition are apparent: the definition is intuitively clear, very general, and, since it uses the finite Shannon sampling series, it is easy to perform analytical calculations, such as taking the derivative. However, this definition also has its drawbacks. For example, questions related to the time concentration behavior cannot easily be answered, and, additionally, it is unclear how this definition is connected to the usual definition of a computable continuous function. In this paper we will answer both questions for signals in $\mathcal{B}_\pi^p$, $1 < p < \infty$. We will also see that for $p = \infty$, i.e., $\mathcal{B}_{\pi,0}^\infty$, the situation is more involved.

Since there are several approaches how to define computable bandlimited signals it is interesting to analyze the connections between these definitions and to study which of the definitions is best suited to work with in certain problems. We will compare different definitions, and analyze whether they lead to the same class of signals. In case they are not equivalent, it is interesting to characterize the differences.

The structure of this paper is as follows. In Section II we discuss the significance of discrete-time and continuous-time signals in signal processing. Then in Sections III, IV, and V we introduce the general notation, the basics of computability theory, and the definition of a computable bandlimited signal, respectively. The time-concentration of signals and first connections to computability are treated in VI. Our main results follow in Sections VII–XII. In Section VII we give a complete characterization of bandlimited locally computable signals with an effectively computable continuous-time concentration. In Section VIII we treat the bandlimited signals with an effectively computable discrete-time concentration. Then, in Section IX we present some consequences of our previous results for the computability of the signal norm. Some aspects of the compiler problem, where one signal representation is to be converted into another representation, are discussed in Section X. A further possibility to define locally computable bandlimited signals is presented in Section XI, and some connections to the previously introduced signal classes are derived. In Section XII we give results for the case of oversampling. Finally, a conclusion is given in Section XIII, where we discuss some open problems.

II. Analog and Digital Domains

Linear time-invariant (LTI) systems are extensively used in signal processing. While most real physical systems are analog and continuous in time, the processing of data is often done on digital devices. Hence, the conversion of signals from the analog to the digital domain and vice versa is essential. The link between both domains is established by various sampling theorems.

A typical procedure how to process an analog signal in the digital domain is illustrated in Fig. 2. First, the continuous-time signal is converted into a discrete-time signal by sampling. Then it is processed by a digital system, which is derived from the actual analog system. Finally, the discrete-time output of the system is converted back into a continuous-time signal. A crucial part of this procedure is the transition from the analog domain into the digital domain and vice versa. The conversion of an analog continuous-time signal into a discrete-time signal can be done by sampling, an operation that, from a theoretical point of view, causes no problems. In contrast, the reconstruction, where it is important that the approximation error can be controlled, can be problematic.

As discussed in the introduction, all bandlimited signals have an infinite duration. Hence, finite impulse response continuous-time LTI systems with bandlimited impulse response do not exist. Nevertheless, the time concentration of the impulse response is an important quantity. In this paper, we consider the Bernstein spaces $\mathcal{B}_\pi^p$, $1 \leq p \leq \infty$, and analyze the computability of their time concentration behavior. We will see that the special cases $\mathcal{B}_\pi^1$ and $\mathcal{B}_{\pi,0}^\infty$, which are important in the theory of bounded input bounded output (BIBO) stable LTI systems, are special in the sense that they exhibit a behavior that is different from the other spaces.

III. Notation

By c_0 we denote the set of all sequences that vanish at infinity. Further, by $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, we denote the usual spaces of pth-power summable sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ with the norm $\|x\|_{\ell^p} = (\sum_{k=-\infty}^{\infty} |x(k)|^p)^{1/p}$, and by $\ell^\infty(\mathbb{Z})$ the space of all bounded sequence with the norm $\|x\|_{\ell^\infty} = \sup_{k\in\mathbb{Z}} |x(k)|$. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \leq p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|f\|_p = \left(\int_\Omega |f(t)|^p \,\mathrm{d}t\right)^{1/p}$ and $L^\infty(\Omega)$ the space of all measurable functions for which the essential supremum norm $\|f\|_\infty = \operatorname{ess\,sup}_{t\in\Omega} |f(t)|$ is finite.

By $\hat{f}$ we denote the Fourier transform of a function f, and by $f|_\mathbb{Z}$ the sequence $\{f(k)\}_{k\in\mathbb{Z}}$, which is the restriction of f to the set $\mathbb{Z}$. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \leq p \leq \infty$, consists of all entire functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [37, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., $\|\cdot\|_{\mathcal{B}_\sigma^p} = \|\cdot\|_p$. A signal in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. $\mathcal{B}_\sigma^2$ is the frequently used space of bandlimited signals with bandwidth σ and finite energy, and $\mathcal{B}_\sigma^\infty$ the space of all bandlimited signals with bandwidth σ that are bounded on the real axis. $\mathcal{B}_{\sigma,0}^\infty$ denotes the space of all signals in $\mathcal{B}_\sigma^\infty$ that vanish on the real axis at infinity. We have $\mathcal{B}_\sigma^r \subsetneq \mathcal{B}_\sigma^s \subsetneq \mathcal{B}_{\sigma,0}^\infty$ for all $1 \leq r < s < \infty$. While the space $\mathcal{B}_\sigma^2$ of bandlimited signals with finite energy has a special physical interpretation, also the extreme cases in the scale of Bernstein spaces, $\mathcal{B}_\sigma^1$ and $\mathcal{B}_\sigma^\infty$, are important. They are, for example, used in system and signal theory to model BIBO stable LTI systems. For every function $h \in \mathcal{B}_\sigma^1$, the convolution integral

$$(Tf)(t) = \int_{-\infty}^{\infty} h(t-\tau) f(\tau) \,\mathrm{d}\tau$$

defines a BIBO stable LTI system $T\colon \mathcal{B}_\sigma^\infty \to \mathcal{B}_\sigma^\infty$.

IV. Basics of Computability

In order to study the question of computability, we introduce some basic notions next. A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. We will not go into details here, for us, it is important that recursive functions are computable by a Turing machine. Details about recursive functions can be found for example in [38].

A set $\mathcal{A} \subseteq \mathbb{N}$ is called recursively enumerable if $\mathcal{A} = \emptyset$ or $\mathcal{A}$ is the range of a recursive function. A set $\mathcal{A} \subseteq \mathbb{N}$ is called recursive if both $\mathcal{A}$ and $\mathbb{N} \setminus \mathcal{A}$ are recursively enumerable.

Definition 1. We say that a set $\mathcal{A} \subsetneq \mathbb{N}$ is a recursively enumerable non-recursive set, if $\mathcal{A}$ is recursively enumerable but not recursive, i.e., if $\mathcal{A}$ is recursively enumerable but $\mathbb{N}\setminus\mathcal{A}$ is not recursively enumerable.

Such recursively enumerable non-recursive sets exist [38, p. 19, 4.4 Proposition] and will be of great importance for the results in this paper. For every recursively enumerable non-recursive set $\mathcal{A} \subsetneq \mathbb{N}$, there exists a recursive enumeration of $\mathcal{A}$, i.e., a recursive function $\phi_\mathcal{A}\colon \mathbb{N} \to \mathcal{A}$ that is surjective and injective.

Alan Turing introduced the concept of a computable real number in [25], [26]. Our definition of a computable real number is based on computable sequences of rational numbers [28, p. 14].

Definition 2. A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and $r_n = (-1)^{s(n)} a(n)/b(n)$, $n \in \mathbb{N}$.

Definition 3. A real number x is said to be computable if there exist a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have $|x - r_n| \leq 2^{-M}$ for all $n \geq \xi(M)$. By $\mathbb{R}_c$ we denote the set of computable real numbers, and by $\mathbb{C}_c = \mathbb{R}_c + i\mathbb{R}_c$ the set of computable complex numbers.

Note that commonly used constants like e and π are computable. A non-computable real number was for example constructed in [39].

Definition 4. A sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is called a computable sequence of computable numbers if there exists a computable double sequence of rationals $\{r_{n,m}\}_{n,m\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that, for all $M \in \mathbb{N}$ and all $n \in \mathbb{N}$, we have $|x_n - r_{n,m}| \leq 2^{-M}$ for all $m \geq \xi(M, n)$.

Note that if a computable sequence of computable numbers $\{x_n\}_{n\in\mathbb{N}}$ converges effectively to a limit x, then x is a computable real number [28, p. 20, Proposition 1]. By effective convergence we mean a convergence where we have an algorithmic control of the approximation error. We will discuss this in more detail at the end of this section.

Definition 5. A sequence $\{x(k)\}_{k\in\mathbb{Z}}$ in c_0 is called computable in c_0 if every number $x(k)$, $k \in \mathbb{Z}$, is computable and there exist a computable sequence $\{x_n\}_{n\in\mathbb{N}} \subset c_0$, where each x_n has only finitely many non-zero elements, and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$ we have $\|x - x_n\|_{\ell^\infty} \leq 2^{-M}$ for all $n \geq \xi(M)$. By $\mathcal{C}c_0$ we denote the set of all sequences that are computable in c_0.

We now give the definition of a computable continuous function on a compact interval [28, p. 25, Definition A(ii)].

Definition 6. Let $I \subset \mathbb{R}$ be an interval, where the endpoints are computable real numbers. A function $f\colon I \to \mathbb{R}$ is called a computable continuous function if

1) f maps every computable sequence $\{t_n\}_{n\in\mathbb{N}} \subset I$ into a computable sequence $\{f(t_n)\}_{n\in\mathbb{N}}$ of computable numbers.
2) there exists a recursive function $d\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ and all $t_1, t_2 \in I$ we have: $|t_1 - t_2| \leq 1/d(M)$ implies $|f(t_1) - f(t_2)| \leq 2^{-M}$.

We next provide the definition of a computable continuous function on $\mathbb{R}$. This definition is a straight forward extension of Definition 6.

Definition 7. A function $f\colon \mathbb{R} \to \mathbb{R}$ is a called a computable continuous function if

1) f maps every computable sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ into a computable sequence $\{f(t_n)\}_{n\in\mathbb{N}}$ of computable numbers.
2) there exists a recursive function $d\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that for all $M, L \in \mathbb{N}$ and all $t_1, t_2 \in [-L, L]$ we have: $|t_1 - t_2| \le 1/d(M,L)$ implies $|f(t_1) - f(t_2)| \le 2^{-M}$.

It can be shown that Definition 7 is equivalent to the following definition that uses sequences of rational polynomials [28, p. 36].

Definition 8. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called a computable continuous function if there exists a computable double sequence of rational polynomials $\{p_{n,L}\}_{n\in\mathbb{N},L\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, such that for all $L, M \in \mathbb{N}$, we have

$$\max_{t\in[-L,L]} |f(t) - p_{n,L}(t)| \le \frac{1}{2^M}$$

for all $n \ge \xi(M, L)$.

Definition 8 has the advantage that it immediately gives us a representation of a computable continuous function that is suited as an input to a Turing machine. According to Definition 8, a computable continuous function is specified by a computable double sequence of rational polynomials $\{p_{n,L}\}_{n\in\mathbb{N},L\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. We call the pair $(\{p_{n,L}\}_{n\in\mathbb{N},L\in\mathbb{N}}, \xi)$ a representation of the function. Note that this representation is not unique.

V. Computable Bandlimited Signals

Next, we define computable bandlimited signals using the same definition as in [31]–[35] that is based on the finite Shannon sampling series as an approximation function.

Definition 9. A function $f \in \mathcal{B}_\sigma^p$, $\sigma > 0$, $p \in [1,\infty]$, is called elementary computable in $\mathcal{B}_\sigma^p$, if there exists a natural number L and a sequence of computable numbers $\{c_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} c_k \frac{\sin\left(\sigma(t - \frac{k\pi}{\sigma})\right)}{\sigma\pi\left(t - \frac{k\pi}{\sigma}\right)}.$$

The building blocks of an elementary computable function are sinc functions. Hence, elementary computable functions are exactly those functions that can be represented by a finite Shannon sampling series with computable coefficients $\{c_k\}_{k=-L}^{L}$. Note that every elementary computable function f is the finite sum of computable continuous functions and hence a computable continuous function. As a consequence, for every $t \in \mathbb{R}_c$ the number $f(t)$ is computable. Further, the sum of finitely many elementary computable functions is elementary computable, as well as the product of an elementary computable function with a computable number.

Definition 10. A signal in $f \in \mathcal{B}_\sigma^p$, $\sigma > 0$, $p \in [1,\infty) \cap \mathbb{R}_c$, is called computable in $\mathcal{B}_\sigma^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}_\sigma^p$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\|f - f_n\|_{\mathcal{B}_\sigma^p} \le \frac{1}{2^M} \tag{1}$$

for all $n \ge \xi(M)$. For $p = \infty$, i.e., $f \in \mathcal{B}_{\sigma,0}^\infty$ we use the analogous definition, where the $\mathcal{B}_\sigma^p$-norm is replaced with the $\mathcal{B}_{\sigma,0}^\infty$-norm. By $\mathcal{CB}_\sigma^p$, $\sigma > 0$, $p \in [1,\infty)\cap\mathbb{R}_c$, we denote the set of all signals in $\mathcal{B}_\sigma^p$ that are computable in $\mathcal{B}_\sigma^p$, and by $\mathcal{CB}_{\sigma,0}^\infty$ the set of all signals in $\mathcal{B}_{\sigma,0}^\infty$ that are computable in $\mathcal{B}_{\sigma,0}^\infty$.

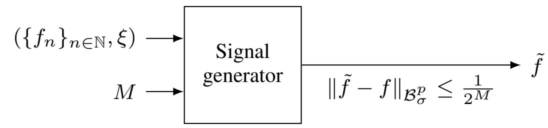

Fig. 3. Signal generator that generates the bandlimited signal $\tilde{f}$ (which is an approximation of f with arbitrarily high accuracy) from the representation $(\{f_n\}_{n\in\mathbb{N}}, \xi)$. M specifies the approximation accuracy.

According to this definition we can approximate any signal $f \in \mathcal{CB}_\sigma^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, by an elementary computable signal, where we have an "effective", i.e. computable control of the approximation error. For every prescribed approximation error $1/2^M$, we can compute an index $M_0 = \xi(M)$ such that the approximation error $\|f - f_n\|_{\mathcal{B}_\pi^p}$ is less than or equal to $1/2^M$ for all $n \ge M_0$. Hence, the type of convergence that we have in (1) is called effective convergence.

We see from Definition 10 that a computable bandlimited signal f in $\mathcal{CB}_\sigma^p$, $\sigma > 0$, $p \in [1,\infty) \cap \mathbb{R}_c$, is specified by a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}_\sigma^p$ and a recursive function ξ. We use the notation $(\{f_n\}_{n\in\mathbb{N}}, \xi)$ to denote this representation of f.

Remark 1. If f is a computable bandlimited signal according to Definition 10, i.e., if $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, or $f \in \mathcal{CB}_{\pi,0}^\infty$, then f is also a computable continuous function according to Definition 7, or, equivalently, Definition 8. The proof of this fact will be given in the Appendix.

In Definition 10 we introduced computable bandlimited signals. The basis for this definition is the time domain behavior of bandlimited signals and Shannon's sampling theorem. We further have seen that for $f \in \mathcal{CB}_\sigma^p$, the pair $(\{f_n\}_{n\in\mathbb{N}}, \xi)$ is a representation of f. Using this representation, the signal f can be approximated, as described in Definitions 9 and 10, up to any given accuracy and generated for example by a signal generator, as depicted in Fig. 3. Note that the operating principle of the signal generator can be analog or digital.

In general, a signal generator obtains some description of the signal to be generated, such as the representation $(\{f_n\}_{n\in\mathbb{N}}, \xi)$ in our example. However, other descriptions are possible and might be preferred by the user depending on the application. These descriptions could include approximations by finite duration splines, signal representations in the frequency domain, or signal representations based on the time domain concentration. If such a different description is given, it needs to be converted into the representation $(\{f_n\}_{n\in\mathbb{N}}, \xi)$ that is understood by the signal generator in our example. Often, it is expected that this conversion can be performed automatically by a program, i.e., a Turing machine. This Turing machine may be seen as a compiler that generates a suitable signal representation for the signal generator. A pivotal question in this context is: for what signal representations is such an algorithmic conversion possible?

In this paper, we have a detailed look on different signal representations and study this question for bandlimited signals. In particular, we consider the signal concentration behavior in continuous time and discrete time. In many applications, for example in signal generators or digital-to-analog converters, the signals are described by their samples, and, therefore, the time concentration behavior is specified in discrete time. A relevant question is whether in this case we can also algorithmically describe the time concentration behavior in continuous time. We will show that this is possible for a large class of signal spaces. However, we will also see that for certain signal spaces, which are important for example in system theory, this problem is not algorithmically solvable.

VI. Time Concentration

Bandlimited signals possess a perfect concentration in the frequency domain in the sense that the Fourier transform of a bandlimited signal is non-zero only on some finite interval. Because of the perfect concentration in the frequency domain, bandlimited signals cannot simultaneously be perfectly concentrated in the time domain.

Next, we return to the question from the introduction and analyze whether it is possible to algorithmically determine the essential time duration of bandlimited signals. For a signal $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, the expression

$$\int_{-L}^{L} |f(t)|^p \, \mathrm{d}t \tag{2}$$

can be considered as a measure of the "amount" of the signal f that is located within the interval $[-L, L]$. Further, the expression

$$\int_{-\infty}^{\infty} |f(t)|^p \, \mathrm{d}t - \int_{-L}^{L} |f(t)|^p \, \mathrm{d}t = \int_{|t|>L} |f(t)|^p \, \mathrm{d}t \tag{3}$$

can be seen as a measure of the concentration of the continuous-time signal f on the time interval $[-L, L]$. The smaller the value, the more concentrated the signal is on the interval. Hence, the study of the time concentration behavior is closely related to the question of how fast the sequence of functions $\{f_L\}_{L\in\mathbb{N}}$, given by

$$f_L(t) = \begin{cases} f(t), & |t| \leq L, \\ 0, & |t| > L, \end{cases}$$

converges to f in the L^p-norm.

For all signals $f \in \mathcal{B}_\pi^p$, the expression in (3) converges to zero as L tends to infinity. The question now is whether, and under what conditions on f, this convergence is effective, i.e., can be described algorithmically.

If $f \in \mathcal{CB}_\pi^p$, $p \in [1, \infty) \cap \mathbb{R}_c$, then there exists a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of elementary computable functions in $\mathcal{B}_\pi^p$ such that $\|f - f_n\|_{\mathcal{B}_\pi^p} \leq 2^{-n}$. It follows that $\big| \|f\|_{\mathcal{B}_\pi^p} - \|f_n\|_{\mathcal{B}_\pi^p} \big| \leq 2^{-n}$, which shows that $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$. Moreover, since the sequence

$$\left\{ \int_{-L}^{L} |f(t)|^p \, \mathrm{d}t \right\}_{L\in\mathbb{N}}$$

is monotonically increasing, the convergence in (3) is effective [28, p. 20, Corollary 2a]. Thus, we have an algorithmic description of the time concentration behavior.

VII. Complete Characterization of Locally Computable Signals

Although the approach taken in Definition 10 is reasonable from an application's perspective, several mathematical problems arise. For example, the Shannon sampling series at Nyquist rate does in general not provide a stable approximation process with norm convergence for signals $f \in \mathcal{B}_\pi^1$ or $f \in \mathcal{B}_{\pi,0}^\infty$ [40]. Further, for $\mathcal{B}_{\pi,0}^\infty$ there exists no stable linear approximation process that uses only the samples $\{f(k)\}_{k\in\mathbb{Z}}$ for the approximation [41]. Moreover, focusing on the local signal behavior in the definition of a computable bandlimited signal may be closer to practical applications, where we often have only a local description of the signals. Hence, we introduce a second very natural definition of a computable bandlimited signal.

The main result of this section is to show that both definitions, i.e. Definition 10 and the definition that will follow, are equivalent for $p \in (1, \infty) \cap \mathbb{R}_c$, and, therefore, describe the same class of computable bandlimited signals.

In the next definition, the concentration of the signal in the time domain is the central property.

Definition 11. We say that a signal $f \in \mathcal{B}_\sigma^p$, $\sigma > 0$, $p \in [1, \infty) \cap \mathbb{R}_c$ has an effectively computable continuous-time concentration if

1) f is a computable continuous function, and
2) there exists a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have
$$\left| \|f\|_{\mathcal{B}_\sigma^p}^p - \int_{-L}^{L} |f(t)|^p \, \mathrm{d}t \right| \leq \frac{1}{2^M} \tag{4}$$
for all $L \geq \xi(M)$.

By $\mathcal{CCB}_\sigma^p$, $\sigma > 0$, $p \in [1, \infty) \cap \mathbb{R}_c$, we denote the set of all signals $f \in \mathcal{B}_\sigma^p$ that have an effectively computable continuous-time concentration. For $p = \infty$, i.e., signals $f \in \mathcal{B}_{\sigma,0}^\infty$, we use an analogous definition, where (4) is replaced by

$$\left| \|f\|_{\mathcal{B}_{\sigma,0}^\infty} - \max_{t\in[-L,L]} |f(t)| \right| \leq \frac{1}{2^M},$$

and denote the set of such functions by $\mathcal{CCB}_{\sigma,0}^\infty$,

Remark 2. Condition 1 in this definition is reasonable, because it ensures that $\int_{-L}^{L} |f(t)|^p \, \mathrm{d}t$ is computable. Condition 2 implies that $\int_{|t|>L} |f(t)|^p \, \mathrm{d}t \leq 2^{-M}$ for all $L \geq \xi(M)$. As discussed before, this is a measure for the time concentration of f.

Remark 3. Condition 2 in Definition 11 implies that $\|f\|_{\mathcal{B}_\sigma^p} \in \mathbb{R}_c$: Since f is a computable continuous function, for $p \in$

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$[1,\infty)\cap\mathbb{R}_c$, the sequence $\{\int_{-L}^{L}|f(t)|^p \,\mathrm{d}t\}_{L\in\mathbb{N}}$ is a computable sequence of computable numbers that converges effectively to $\|f\|_{\mathcal{B}_\sigma^p}^p$. Hence, $\|f\|_{\mathcal{B}_\sigma^p}$ is a computable number. The same is true for $p=\infty$ with the usual modifications.

Definition 11, which employs the concept of a computable continuous function and the effective approximation of the norm, gives us more flexibility to represent a function f, compared to Definition 10, where only finite Shannon sampling series are allowed for the approximation. In particular, the building blocks of the approximation do not have to be in $\mathcal{B}_\pi^p$. This flexibility will be used in Section XII to study oversampling.

According to Definition 11, a function $f\in\mathcal{CCB}_\pi^p$ is described by two algorithms. The first computes for a given input $L\in\mathbb{R}_c$ the representation of f on the interval $[-L,L]$. We denote this program by Φ_f. The second computes for an input $M\in\mathbb{N}$ the time concentration according to (4). We denote this program by ξ. Hence, a signal $f\in\mathcal{CCB}_\pi^p$ is completely described by the pair of programs (Φ_f,ξ).

The next theorem shows that the sets $\mathcal{CCB}_\pi^p$ and $\mathcal{CB}_\pi^p$ coincide for $p\in(1,\infty)\cap\mathbb{R}_c$.

Theorem 1. *Let $p\in(1,\infty)\cap\mathbb{R}_c$. Then we have $\mathcal{CB}_\pi^p=\mathcal{CCB}_\pi^p$.*

Theorem 1 gives us a simple characterization of $\mathcal{CB}_\pi^p$ signals. Note that such a characterization is very useful, because for a given signal $f\in\mathcal{B}_\pi^p$ it is sometimes difficult to check whether $f\in\mathcal{CB}_\pi^p$ or not, by using the definition of $\mathcal{CB}_\pi^p$.

Before we prove Theorem 1, we give a result for the cases $p=1$ and $p=\infty$ which have been excluded in Theorem 1.

Theorem 2. *For $p=1$ and $p=\infty$, we have $\mathcal{CB}_\pi^1\subseteq\mathcal{CCB}_\pi^1$ and $\mathcal{CB}_{\pi,0}^\infty\subsetneq\mathcal{CCB}_{\pi,0}^\infty$, respectively.*

Remark 4. Whether $\mathcal{CB}_\pi^1$ is a proper subset of $\mathcal{CCB}_\pi^1$ is an open problem.

Proof of Theorem 1. We first show that $\mathcal{CB}_\pi^p\supseteq\mathcal{CCB}_\pi^p$. Let $p\in(1,\infty)\cap\mathbb{R}_c$ and $f\in\mathcal{CCB}_\pi^p$ be arbitrary. For $N\in\mathbb{N}$, we consider

$$F_N(t)=(P_Nf)(t)=\int_{-N}^{N}f(\tau)\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\,\mathrm{d}\tau. \quad (5)$$

The integral in (5) is computable for all $t\in\mathbb{R}_c$, because f is a computable continuous function on $[-N,N]$ [28, Theorem 5, p. 35]. For $p\in(1,\infty)$, $P_N\colon L^p(\mathbb{R})\to L^p(\mathbb{R})$ is a bounded linear operator, and we have $\|P_N\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\le C_1(p)$, where the constant $C_1(p)$ is independent of N [42, p. 256]. It follows that $F_N\in L^p(\mathbb{R})$. Let

$$g_N(t)=\begin{cases} f(t), & |t|\le N,\\ 0, & |t|>N.\end{cases}$$

Since f is continuous on $[-N,N]$, we see that $g_N\in L^1(\mathbb{R})$. It follows that

$$F_N(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{g}_N(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega,$$

according to the convolution theorem of the Fourier transform. This shows that $F_N\in\mathcal{B}_\pi^p$.

Since f is a computable continuous function on $[-N,N]$, it follows that $\{F_N(k)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers. For $|t|\ge N+1$ we have

$$\begin{aligned}|F_N(t)|&\le\int_{-N}^{N}|f(\tau)|\left|\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\right|\mathrm{d}\tau\\ &\le\frac{1}{|t|-N}\int_{-N}^{N}|f(\tau)|\,\mathrm{d}\tau. \qquad (6)\end{aligned}$$

For $M>N$, let

$$(S_MF_N)(t)=\sum_{k=-M}^{M}F_N(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

For $M>N$ and $K\in\mathbb{N}$, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}|(S_{M+K}F_N)(t)-(S_MF_N)(t)|^p\,\mathrm{d}t\\ &\le C_2(p)\sum_{M+1\le|k|\le M+K}|F_N(k)|^p\\ &\le\left(\int_{-N}^{N}|f(\tau)|\,\mathrm{d}\tau\right)^p C_2(p)\sum_{M+1\le|k|\le M+K}\frac{1}{(|k|-N)^p}\\ &=2\left(\int_{-N}^{N}|f(\tau)|\,\mathrm{d}\tau\right)^p C_2(p)\sum_{k=M+1}^{M+K}\frac{1}{(k-N)^p}, \qquad (7)\end{aligned}$$

where we used the Plancherel–Pólya inequality [43, p. 152] in the first inequality and (6) in the second. Further, we have

$$\begin{aligned}\sum_{k=M+1}^{M+K}\frac{1}{(k-N)^p}&=\sum_{k=M+1-N}^{M+K-N}\frac{1}{k^p}\\ &<\sum_{k=M+1-N}^{M+K-N}\int_{k-1}^{k}\frac{1}{\tau^p}\,\mathrm{d}\tau=\int_{M-N}^{M+K-N}\frac{1}{\tau^p}\,\mathrm{d}\tau\\ &=\frac{1}{p-1}\left(\frac{1}{(M-N)^{p-1}}-\frac{1}{(M+K-N)^{p-1}}\right)\\ &<\frac{1}{(p-1)(M-N)^{p-1}}. \qquad (8)\end{aligned}$$

Combining (7) and (8), we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}|(S_{M+K}F_N)(t)-(S_MF_N)(t)|^p\,\mathrm{d}t\\ &\quad<2\left(\int_{-N}^{N}f(\tau)\,\mathrm{d}\tau\right)^p\frac{C_2(p)}{(p-1)(M-N)^{p-1}}\end{aligned}$$

for all $M>N$ and $K\in\mathbb{N}$ arbitrary. This shows that $\{S_MF_N\}_{M\in\mathbb{N}}$ is an effective Cauchy sequence that converges to F_N in the L^p-norm. Hence, the convergence is effective, and it follows that $F_N\in\mathcal{CB}_\pi^p$. Since $f\in\mathcal{CCB}_\pi^p\subsetneq\mathcal{B}_\pi^p$, we have

$$f(t)=(Pf)(t)=\int_{-\infty}^{\infty}f(\tau)\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\,\mathrm{d}\tau,\quad t\in\mathbb{R},$$

and consequently

$$f(t)-F_N(t)=\int_{|\tau|>N}f(\tau)\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\,\mathrm{d}\tau,\quad t\in\mathbb{R}.$$

Let

$$h_N(t) = \begin{cases} 0, & |t| \leq N, \\ f(t), & |t| > N. \end{cases}$$

Then we have

$$\begin{aligned} \|f - F_N\|_{\mathcal{B}_\pi^p} &= \|Ph_N\|_{\mathcal{B}_\pi^p} \leq C_1(p)\|h_N(t)\|_p \\ &= C_1(p)\left(\|f\|_{\mathcal{B}_\pi^p}^p - \int_{-N}^{N} |f(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}, \end{aligned}$$

where the content of the parenthesis in the last line converges effectively to zero because $f \in \mathcal{CCB}_\pi^p$. Thus, $\{F_N\}_{N\in\mathbb{N}}$ converges effectively to f in the $\mathcal{B}_\pi^p$-norm. This shows that $f \in \mathcal{CB}_\pi^p$, and consequently that $\mathcal{CB}_\pi^p \supseteq \mathcal{CCB}_\pi^p$.

It remains to show that $\mathcal{CB}_\pi^p \subseteq \mathcal{CCB}_\pi^p$. Let $p \in [1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{CB}_\pi^p$ be arbitrary. Since $f \in \mathcal{CB}_\pi^p$, there exists a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of elementary computable functions, such that

$$\|f - f_n\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Let $M \in \mathbb{N}$ be arbitrary. We choose $n_1 = M+1$. Since f_{n_1} is a finite Shannon sampling series, there exists a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that

$$\left(\int_{|t|>N} |f_{n_1}(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \leq \frac{1}{2^{M+1}}$$

for all $N \geq \xi(M)$. Hence, for $N \in \mathbb{N}$, we have

$$\begin{aligned} &\left(\|f\|_{\mathcal{B}_\pi^p}^p - \int_{-N}^{N} |f(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_{|t|>N} |f(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq \left(\int_{|t|>N} |f(t) - f_{n_1}(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_{|t|>N} |f_{n_1}(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq \frac{1}{2^{n_1}} + \left(\int_{|t|>N} |f_{n_1}(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}$$

Thus, for $N \geq \xi(M)$, we obtain

$$\left(\|f\|_{\mathcal{B}_\pi^p}^p - \int_{-N}^{N} |f(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \leq \frac{1}{2^{M+1}} + \frac{1}{2^{M+1}} = \frac{1}{2^M},$$

and consequently

$$\|f\|_{\mathcal{B}_\pi^p}^p - \int_{-N}^{N} |f(t)|^p \, \mathrm{d}t \leq \frac{1}{2^{pM}} < \frac{1}{2^M}, \tag{9}$$

which shows that item 2 of Definition 11 is satisfied. Further, item 1 is satisfied, because $f \in \mathcal{CB}_\pi^p$. Hence, we have shown that $f \in \mathcal{CCB}_\pi^p$. Note that the previous calculation is valid also for $p = 1$. Thus, we have $\mathcal{CCB}_\pi^1 \supseteq \mathcal{CB}_\pi^1$, which proves the $p = 1$ case of Theorem 2. $\square$

Proof of Theorem 2. The proof for $p = 1$ has already been given in the proof of Theorem 1.

It remains to prove the case $p = \infty$. Let $f \in \mathcal{CB}_{\pi,0}^\infty$. In [44] it has been shown that if $f \in \mathcal{CB}_{\pi,0}^\infty$ then there exists a recursive function $\eta\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$|f(t)| \leq \frac{1}{2^M}$$

for all $t \in \mathbb{R}$ with $|t| \geq \eta(M)$. It follows for

$$g_L(t) = \begin{cases} f(t), & |t| \leq L, \\ 0, & |t| > L, \end{cases}$$

that

$$\begin{aligned} \left| \|f\|_{\mathcal{B}_{\pi,0}^\infty} - \max_{t\in[-L,L]} |f(t)| \, \mathrm{d}t \right| &= \left| \|f\|_{\mathcal{B}_{\pi,0}^\infty} - \|g_L\|_\infty \right| \\ &\leq \|f - g_L\|_\infty = \sup_{|t|>L} |f(t)| \leq \frac{1}{2^M} \end{aligned}$$

for all $L \geq \eta(M)$. Thus, item 2 of Definition 11 is satisfied. Item 1 is satisfied, because $f \in \mathcal{CB}_{\pi,0}^\infty$. Hence, we have shown that $f \in \mathcal{CCB}_{\pi,0}^\infty$, and consequently that $\mathcal{CB}_{\pi,0}^\infty \subseteq \mathcal{CCB}_{\pi,0}^\infty$. The fact that $\mathcal{CB}_{\pi,0}^\infty$ is a proper subset of $\mathcal{CCB}_{\pi,0}^\infty$ will be proved in Remark 6. $\square$

As we have seen, there are different approaches to define computable bandlimited signals. It is interesting to further explore the similarities and differences of these definitions. In particular, it would be desirable to identify a simple approach that is well suited to verify computability in practical applications.

Theorem 3. *Let $p \in (1,\infty) \cap \mathbb{R}_c$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if*

1) *$f \in \mathcal{B}_\pi^p$,*
2) *f is a computable continuous function, and*
3) *$\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$.*

For many practically relevant problems, we know that the involved signals $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$, are computable continuous functions. Hence, as soon as we know that $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$, we can conclude that $f \in \mathcal{CB}_\pi^p$.

Proof of Theorem 3. Let $p \in (1,\infty) \cap \mathbb{R}_c$. "$\Rightarrow$": Let $f \in \mathcal{CB}_\pi^p$. Then there exists a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of elementary computable functions such that

$$\|f - f_n\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Since f_n is a finite Shannon sampling series, we have $\|f_n\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$. Hence, it follows that

$$\left| \|f\|_{\mathcal{B}_\pi^p} - \|f_n\|_{\mathcal{B}_\pi^p} \right| \leq \|f - f_n\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^n},$$

i.e., $\|f\|_{\mathcal{B}_\pi^p}$ is the limit of a computable sequence of computable numbers that converges effectively. This implies that $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$ [28, p. 20, Proposition 1]. Since $f \in \mathcal{CB}_\pi^p$, f is also a computable continuous function. This completes the first part of the proof.

"$\Leftarrow$": Let $f \in \mathcal{B}_\pi^p$ be a computable continuous function with $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$. For $N \in \mathbb{N}$, we set

$$I_N = \int_{-N}^{N} |f(f)|^p \, \mathrm{d}t.$$

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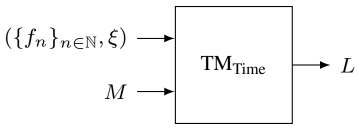

Fig. 4. Turing machine $\mathrm{TM}_{\mathrm{Time}}$ that computes a time instant L such that the signal f is essentially concentrated on the interval $[-L, L]$.

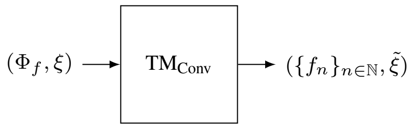

Fig. 5. Turing machine $\mathrm{TM}_{\mathrm{Conv}}$ that converts any representation (Φ_f, ξ) of $f \in \mathcal{CCB}_\pi^p$ into a representation $(\{f_n\}_{n\in\mathbb{N}}, \tilde{\xi})$ of a computable bandlimited signal in $\mathcal{CB}_\pi^p$.

Since f is a computable continuous function, it follows that $\{I_N\}_{N\in\mathbb{N}}$ is a monotonically increasing computable sequence of computable numbers. Its limit $\|f\|_{\mathcal{B}_\pi^p}^p$ is a computable number, because $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$. Since a monotonically increasing computable sequence of computable numbers that converges to a computable number, converges effectively [28, p. 20, Corollary 2a], we see that $f \in \mathcal{CCB}_\pi^p$. From Theorem 1 it follows that $f \in \mathcal{CB}_\pi^p$. $\square$

In the proof of Theorem 1, we have seen in (9) that the time concentration can be algorithmically determined, i.e., computed on a Turing machine. In other words, there exists a Turing machine $\mathrm{TM}_{\mathrm{Time}}$ with inputs $M \in \mathbb{N}$ and $f \in \mathcal{CB}_\pi^p$, where f is represented by the pair $(\{f_n\}_{n\in\mathbb{N}}, \xi)$, that computes a time instant $L \in \mathbb{R}_c$ such that

$$\left| \|f\|_{\mathcal{B}_\pi^p}^p - \int_{-L}^{L} |f(t)|^p \, \mathrm{d}t \right| \leq \frac{1}{2^M}.$$

This Turing machine is depicted in Fig. 4.

The proof of Theorem 1 gives us also the existence of a Turing machine $\mathrm{TM}_{\mathrm{Conv}}$ that takes a signal $f \in \mathcal{CCB}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$, represented by (Φ_f, ξ), and outputs a representation $(\{f_n\}_{n\in\mathbb{N}}, \tilde{\xi})$ of a computable bandlimited signal in $\mathcal{CB}_\pi^p$.

The results of this section, in particular the theorems and the existence of the Turing machines $\mathrm{TM}_{\mathrm{Time}}$ and $\mathrm{TM}_{\mathrm{Conv}}$, show that, for $p \in (1,\infty) \cap \mathbb{R}_c$, both approaches to define computability for signals in $\mathcal{B}_\pi^p$ (Definitions 10 and 12 are) valid and useful.

VIII. Local Computability and the Discrete ℓ^p-Norm

Our goal in this section is to continue the analysis of the time concentration behavior, only now in discrete time.

We first present an example which illustrates the difference between time concentration in discrete time and continuous time. Already very simple signals show that a good concentration behavior in discrete time does not necessarily lead to a good concentration in continuous time. Consider, for example, the signal

$$f_1(t) = \frac{\sin(\pi t)}{\pi t}.$$

We have

$$f_1(k) = \begin{cases} 1, & k = 0, \\ 0, & k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

This shows that f_1 has an excellent discrete-time concentration behavior. The concentration can be measured for example in the ℓ^1-norm, which plays an important role in the theory of BIBO stable LTI systems. However, we have $f_1 \notin \mathcal{B}_\pi^1$, because the concentration in continuous time is too weak for the signal to be absolutely integrable.

For $N \in \mathbb{N}$, let

$$g_N(t) = \frac{1}{C(N)} \sum_{k=1}^{N} (-1)^k \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

with

$$C(N) = -\frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k - \frac{1}{2}}.$$

The signal g_N has interesting properties. For any $\delta > 0$, we can find an $N_0 \in \mathbb{N}$ such that $|g_N(k)| \leq \delta$ for all $k \in \mathbb{Z}$ and all $N \geq N_0$. On the other hand, we have $|g_N(N + 1/2)| = 1$ for all $N \in \mathbb{N}$. Hence, for N large enough, the signal values of g_N in discrete time can be made arbitrarily small, whereas in continuous time this is not the case. It is interesting to note that this phenomenon implies that we cannot always algorithmically compute the continuous-time concentration from the discrete-time signal behavior, as we will see.

Definition 12. We say that a signal $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$ has an effectively computable discrete-time concentration if

1) f is a computable continuous function, and
2) there exists a recursive function $\xi \colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\left| \|f|_{\mathbb{Z}}\|_{\ell^p}^p - \sum_{k=-N}^{N} |f(k)|^p \right| \leq \frac{1}{2^M} \tag{10}$$

for all $N \geq \xi(M)$.

By $\mathcal{DCB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, we denote the set of all signals $f \in \mathcal{B}_\pi^p$ that have an effectively computable discrete-time concentration. For $p = \infty$, i.e., signals $f \in \mathcal{B}_{\pi,0}^\infty$, we use an analogous definition, where (10) is replaced by

$$\left| \|f|_{\mathbb{Z}}\|_{\ell^\infty} - \max_{\substack{k\in\mathbb{Z},\\ |k|\leq N}} |f(k)| \right| \leq \frac{1}{2^M},$$

and denote the set of such functions by $\mathcal{DCB}_{\pi,0}^\infty$.

Remark 5. Condition 2 in Definition 12 implies that $\|f|_{\mathbb{Z}}\|_{\ell^p} \in \mathbb{R}_c$: Since f is a computable continuous function, for $p \in [1,\infty) \cap \mathbb{R}_c$, the sequence $\{\sum_{k=-L}^{L} |f(k)|^p \ \mathrm{d}t\}_{L\in\mathbb{N}}$ is a computable sequence of computable numbers that converges effectively to $\|f|_{\mathbb{Z}}\|_{\ell^p}^p$. Hence, $\|f|_{\mathbb{Z}}\|_{\ell^p}$ is a computable number. The same is true for $p = \infty$ with the usual modifications.

In Theorem 1 we provided, for $p \in (1,\infty) \cap \mathbb{R}_c$, a characterization of computable bandlimited signals using the continuous-time concentration behavior. In the next theorem we give the analogous result for the discrete-time concentration behavior.

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Theorem 4. *Let $p \in (1,\infty)\cap\mathbb{R}_c$. Then we have $\mathcal{CB}_\pi^p = \mathcal{DCB}_\pi^p$.*

Proof. We first show that $\mathcal{CB}_\pi^p \subseteq \mathcal{DCB}_\pi^p$. Let $p \in (1,\infty)\cap\mathbb{R}_c$ and $f \in \mathcal{CB}_\pi^p$ be arbitrary. Then f is a computable continuous function according to Remark 1. Further, let

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)},$$

then $\{S_N f\}_{N\in\mathbb{N}}$ converges effectively to f in the $\mathcal{B}_\pi^p$-norm [35, Theorem 2]. We have

$$\begin{aligned}\|f|_{\mathbb{Z}}\|_{\ell^p}^p - \sum_{k=-N}^{N}|f(k)|^p &= \sum_{|k|>N}|f(k)|^p\\ &= \sum_{k=-\infty}^{\infty}|f(k)-(S_N f)(k)|^p\\ &\leq (1+\pi)^p\|f-S_N f\|_{\mathcal{B}_\pi^p}^p,\end{aligned}$$

where we used Nikol'skiĭ's inequality [37, p. 49] in the last inequality. This shows that item 2 of Definition 12 is satisfied.

We now show that $\mathcal{CB}_\pi^p \supseteq \mathcal{DCB}_\pi^p$. Let $p \in (1,\infty)\cap\mathbb{R}_c$ and $f \in \mathcal{DCB}_\pi^p$ be arbitrary. Since f is a computable continuous function, it follows from the definition that $\{f(k)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers, and, as a consequence, $\{S_N f\}_{N\in\mathbb{N}}$ is a computable sequence of elementary computable functions in $\mathcal{B}_\pi^p$. Further, we have

$$\begin{aligned}\|f-S_N f\|_{\mathcal{B}_\pi^p}^p &\leq C_2(p)\sum_{k=-\infty}^{\infty}|f(k)-(S_N f)(k)|^p\\ &= C_2(p)\sum_{|k|>N}|f(k)|^p\\ &= C_2(p)\left(\|f|_{\mathbb{Z}}\|_{\ell^p}^p - \sum_{k=-N}^{N}|f(k)|^p\right),\end{aligned}$$

where we used the Plancherel–Pólya inequality [43, p. 152] in the first inequality. Thus, $\{S_N f\}_{N\in\mathbb{N}}$ converges effectively to f in the $\mathcal{B}_\pi^p$-norm, which implies that $f \in \mathcal{CB}_\pi^p$. □

Combining Theorems 1 and 4 immediately gives the next corollary.

Corollary 1. *Let $p \in (1,\infty)\cap\mathbb{R}_c$. Then we have $\mathcal{DCB}_\pi^p = \mathcal{CCB}_\pi^p$.*

Corollary 1 couples the discrete-time behavior with the continuous-time behavior of the signal. Theorem 4 and Corollary 1 show that, for $p \in (1,\infty)\cap\mathbb{R}_c$, Definitions 10, 11, and 12 all lead to the same set of computable bandlimited signals. For $p=1$ and $p=\infty$ this is no longer the case. Hence, for the practical relevant spaces $\mathcal{B}_\pi^1$ and $\mathcal{B}_{\pi,0}^\infty$, we can, in general, not infer the computability of the continuous-time signal from a good computability behavior of the discrete-time signal.

Theorem 5. *We have $\mathcal{CB}_\pi^1 \subsetneq \mathcal{DCB}_\pi^1$ and $\mathcal{CB}_{\pi,0}^\infty \subsetneq \mathcal{DCB}_{\pi,0}^\infty$.*

Proof. The fact that $\mathcal{CB}_\pi^1 \subsetneq \mathcal{DCB}_\pi^1$ is a direct consequence of [35, Corollary 1].

Next, we prove that $\mathcal{CB}_{\pi,0}^\infty \subsetneq \mathcal{DCB}_{\pi,0}^\infty$. For the proof we employ the same function f_3 that was used in the proof of Corollary 2 in [44]. We review its construction. For $n \in \mathbb{N}$, let

$$g_n(t) = \sum_{k=0}^{2n}(-1)^k\frac{\sin(\pi(t-k))}{\pi(t-k)},$$

and

$$C(n) = g_n(2n+1/2) = \frac{1}{\pi}\sum_{k=0}^{2n}\frac{1}{2n+\frac{1}{2}-k}.$$

Further, let $\mathcal{A} \subsetneq \mathbb{N}$ be an arbitrary recursively enumerable non-recursive set, and $\phi_{\mathcal{A}}\colon \mathbb{N}\to\mathcal{A}$ a recursive enumeration of $\mathcal{A}$, where $\phi_{\mathcal{A}}$ is a bijection. Then f_3 is defined as

$$f_3(t) = \sum_{n=1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(n)}}\frac{g_n(t-N_n)}{C(n)},$$

where $\{N_n\}_{n\in\mathbb{N}}$ is a strictly monotonically increasing computable sequence of even numbers. We can assume that $N_n > n^2$. For more details about the construction of $\{N_n\}_{n\in\mathbb{N}}$, see [44]. Note that $f_3 \in \mathcal{B}_{\pi,0}^\infty$ and $f_3|_{\mathbb{Z}} \in \mathcal{C}c_0$, according to [44, Corollary 2].

In [44, Remark 6] it was shown that $f_3 \notin \mathcal{CB}_{\pi,0}^\infty$. We will prove next that $f \in \mathcal{DCB}_{\pi,0}^\infty$, by verifying that items 1 and 2 of Definition 12 are satisfied.

We start with item 1, i.e., we prove that f_3 is a computable continuous function on $\mathbb{R}$. To this end, we will check that f_3 satisfies all conditions of Definition 8. For $n \geq 2$ and $t \leq -n$ we have

$$\begin{aligned}|g_n(t)| &\leq \frac{1}{\pi}\sum_{k=0}^{2n}\frac{1}{|t-k|} \leq \frac{1}{\pi}\sum_{k=0}^{2n}\frac{1}{n-k}\\ &< \frac{1}{\pi}\sum_{k=0}^{2n}\int_{k-1}^{k}\frac{1}{n+\tau}\,\mathrm{d}\tau = \frac{1}{\pi}\int_{-1}^{2n}\frac{1}{n+\tau}\,\mathrm{d}\tau\\ &= \frac{1}{\pi}\log\left(\frac{3n}{n-1}\right) \leq \frac{1}{\pi}\log(6).\end{aligned}$$

Let $L \in \mathbb{N}$, $L \geq 2$ be arbitrary. For $n > L$ and $t \in [-L,L]$ we have

$$t - N_n \leq L - N_n < n - N_n < n - n^2 < -n,$$

and consequently

$$|g_n(t-N_n)| < \frac{1}{\pi}\log(6).$$

Let

$$q_m(t) = \sum_{n=1}^{m}\frac{1}{2^{\phi_{\mathcal{A}}(n)}}\frac{g_n(t-N_n)}{C(n)}.$$

It follows that

$$\begin{aligned}|f_3(t)-q_m(t)| &\leq \sum_{n=m+1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(n)}C(n)}|g_n(t-N_n)|\\ &< \frac{1}{\pi}\log(6)\sum_{n=m+1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(n)}C(n)}\\ &< \frac{1}{\pi}\frac{\log(6)}{C(m+1)}\sum_{n=m+1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(n)}}\\ &\leq \frac{1}{\pi}\frac{\log(6)}{C(m+1)}\end{aligned}$$

$p=1$	$p \in (1,\infty)\cap\mathbb{R}_c$	$p=\infty$
$\mathcal{CB}_\pi^1 \subseteq \mathcal{CCB}_\pi^1$ $\mathcal{CB}_\pi^1 \subsetneq \mathcal{DCB}_\pi^1$	$\mathcal{CB}_\pi^p = \mathcal{DCB}_\pi^p = \mathcal{CCB}_\pi^p$	$\mathcal{CB}_{\pi,0}^\infty \subsetneq \mathcal{CCB}_{\pi,0}^\infty$ $\mathcal{CB}_{\pi,0}^\infty \subsetneq \mathcal{DCB}_{\pi,0}^\infty$
Thm. 2, 5	Thm. 1, 4, and Cor. 1	Thm. 2, 5

Fig. 6. Summary of the main findings of Sections VII and VIII.

for all $t \in [-L, L]$ and $m \geq L$. Hence, we see that the computable sequence $\{q_m\}_{m\in\mathbb{N}}$ converges effectively to f_3 on $[-L, L]$. In our calculation, the convergence speed depends recursively on L. Since q_m is a finite sum of sinc functions with computable coefficients, we can approximate each q_m on the interval $[-L, L]$ by the Taylor polynomial $Q_{m,L}$ of q_m with sufficiently high degree, such that

$$|q_m(t) - Q_{m,L}(t)| \leq \frac{1}{2^m}$$

for all $t \in [-L, L]$. Note that $\{Q_{m,L}\}$ is a commutable double sequence of polynomials. We have

$$\begin{aligned}|f_3(t) - Q_{m,L}(t)| &\leq |f_3(t) - q_m(t)| + |q_m(t) - Q_{m,L}(t)| \\ &< \frac{1}{\pi}\frac{\log(6)}{C(m+1)} + \frac{1}{2^m} < \frac{2}{C(m+1)} + \frac{1}{2^m}\end{aligned}$$

for all $t \in [-L, L]$. Thus, f_3 is a computable continuous function according to Definition 8. Hence, item 1 of Definition 12 is satisfied.

Since $f_3|_{\mathbb{Z}} \in \mathcal{C}c_0$, it follows that $\|f_3|_{\mathbb{Z}}\|_{\ell^\infty}$ is a computable number. Further, $\{\max_{|k|\leq N}|f_3(k)|\}_{N\in\mathbb{N}}$ is a monotonically increasing sequence of computable numbers. Hence, according to [28, p. 20, Corollary 2a], the convergence of

$$\left| \|f_3|_{\mathbb{Z}}\|_{\ell^\infty} - \max_{\substack{k\in\mathbb{Z},\\ |k|\leq N}} |f_3(k)| \right|$$

is effective, and item 2 of Definition 12 is satisfied. □

We summarize some of the main findings of Sections VII and VIII in Fig. 6.

IX. Computability of the Norm

In Remarks 3 and 5 we have seen that the computability of the norms $\|f\|_{\mathcal{B}_\sigma^p}$ and $\|f|_{\mathbb{Z}}\|_{\ell^p}$ is of high relevance. In this section we further discuss the computability of these norms.

1) For every $f \in \mathcal{B}_\sigma^p$, $p \in [1,\infty)\cap\mathbb{R}_c$, that is additionally a computable continuous function, we have $f \in \mathcal{CCB}_\sigma^p$ if and only if $\|f\|_{\mathcal{B}_\sigma^p} \in \mathbb{R}_c$. That is, for all functions $f \in \mathcal{B}_\sigma^p$, $p \in [1,\infty)\cap\mathbb{R}_c$, that satisfy condition 1 in Definition 11, condition 2 in Definition 11 is equivalent to $\|f\|_{\mathcal{B}_\sigma^p} \in \mathbb{R}_c$. The "$\Rightarrow$" direction of this equivalence was discussed in Remark 3 and the "$\Leftarrow$" direction was treated in [35].
2) For every $f \in \mathcal{B}_{\sigma,0}^\infty$ that is additionally a computable continuous function, we have $\lim_{|t|\to\infty}|f(t)| = 0$. Hence, there exists a natural number N_0 such that $\|f\|_{\mathcal{B}_{\sigma,0}^\infty} = \max_{|t|\leq N_0}|f(t)|$, and, consequently, we have $\|f\|_{\mathcal{B}_{\sigma,0}^\infty} \in \mathbb{R}_c$ [28, p. 40, Theorem 7]. This shows that, for $p=\infty$, condition 2 in Definition 11 is expendable.
3) The statement of item 1 in this list is also true for Definition 12: For all functions $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty)\cap\mathbb{R}_c$, that satisfy condition 1 in Definition 12, condition 2 in Definition 12 is equivalent to $\|f|_{\mathbb{Z}}\|_{\ell^p} \in \mathbb{R}_c$.
4) The statement of item 2 in this list is also true for Definition 12. That is, for $p = \infty$, condition 2 in Definition 12 is expendable.

Regarding the condition $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$, we see a very interesting difference between $p \in [1,\infty)\cap\mathbb{R}_c$ and $p=\infty$. For computable continuous functions $f \in \mathcal{B}_{\pi,0}^\infty$, we always have $\|f\|_{\mathcal{B}_{\pi,0}^\infty} \in \mathbb{R}_c$. In contrast, for $p \in [1,\infty)\cap\mathbb{R}_c$, as it will become clear from the proof of Theorem 6, there exist computable continuous functions $f \in \mathcal{B}_\pi^p$ for which we have $\|f\|_{\mathcal{B}_\pi^p} \notin \mathbb{R}_c$.

Remark 6. Items 1 and 4 imply that $\mathcal{DCB}_{\pi,0}^\infty = \mathcal{CCB}_{\pi,0}^\infty$. From Theorem 5 we know that $\mathcal{CB}_{\pi,0}^\infty \subsetneq \mathcal{DCB}_{\pi,0}^\infty$. Hence, we also have $\mathcal{CB}_{\pi,0}^\infty \subsetneq \mathcal{CCB}_{\pi,0}^\infty$.

Items 2 and 4 in the above discussion can, in general, not be exploited for solving practical problems algorithmically. Consider, for example, the problem of computing the peak value of computable continuous signals in $\mathcal{B}_{\pi,0}^\infty$. We can ask: Does there exist a Turing machine that for every input $f \in \mathcal{B}_{\pi,0}^\infty$, described as a computable continuous function, is able to compute $\|f\|_{\mathcal{B}_{\pi,0}^\infty}$? In [44] it has been shown that such a Turing machine does not exist.

X. Compiler Problem

In this section we come back to the problem of automatically converting one signal representation into another.

Let $f \in \mathcal{DCB}_{\pi,0}^\infty$, and assume that we have a representation of f according to Definition 12, i.e., we have an algorithm that computes the values of f as a computable continuous function and an algorithm that describes the discrete-time concentration of f on $\mathbb{Z}$, as in condition 2 in Definition 12. The goal is to algorithmically compute the continuous-time concentration behavior of f on $\mathbb{R}$ in the sense of condition 2 in Definition 11. In general, this is not possible. That is, there exists no Turing machine that performs this computation for arbitrary inputs $f \in \mathcal{DCB}_{\pi,0}^\infty$ [44].

Clearly, for every $f \in \mathcal{DCB}_{\pi,0}^\infty$ there exists a recursive function ξ that describes the concentration behavior according to condition 2 in Definition 11. However, there exists no Turing machine that always can compute this function from a representation of f according to Definition 12.

On the positive side, as we have shown, for $f \in \mathcal{DCB}_\pi^p$, $p \in (1,\infty)\cap\mathbb{R}_c$, and in particular for computable bandlimited signals with finite energy, this conversion can always be performed algorithmically.

Assume that we have a computable continuous function f. As we have seen, if $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty)\cap\mathbb{R}_c$, then we can without problems switch between computability on $\mathbb{R}$ and computability on $\mathbb{Z}$. In contrast, for $\mathcal{B}_{\pi,0}^\infty$ this is not possible, although a computability representation on $\mathbb{Z}$ contains all information about the signal, as we will discuss next. For every $f \in \mathcal{CB}_{\pi,0}^\infty$, there exists a Turing machine $\mathrm{TM}_{f,\mathbb{Z}}$ that for every input $k \in \mathbb{Z}$ computes $f(k) = \mathrm{TM}_{f,\mathbb{Z}}(k)$. That is, there exists a Turing machine that for all inputs $(k, M) \in \mathbb{N}^2$ computes a rational number $r_{M,k}$ such that $|f(k) - r_{M,k}| \leq 2^{-M}$. Further, the signal f is uniquely determined by the samples $\{f(k)\}_{k\in\mathbb{Z}}$, i.e., $f \in \mathcal{B}_{\pi,0}^{\infty}$ with $f(k) = 0$ for all $k \in \mathbb{Z}$, implies $f(t) = 0$ for all $t \in \mathbb{R}$. Hence, the above description of the signal contains all information about the signal. However, from this complete information, we cannot always compute a description of the continuous time behavior as in condition 2 in Definition 11.

XI. A Further Possibility to Define Locally Computable Bandlimited Signals

In [44] a further definition for a locally computable bandlimited signal in $\mathcal{B}_{\pi,0}^{\infty}$ was given. This definition is easily generalized to signals $f \in \mathcal{B}_{\pi}^{p}$, $p \in [1,\infty) \cap \mathbb{R}_c$.

Definition 13. We call a signal $f \in \mathcal{B}_{\pi}^{p}$, $p \in [1,\infty) \cap \mathbb{R}_c$, locally computable, if there exist a computable double sequence of computable continuous functions $\{f_{n,L}\}_{n\in\mathbb{N},L\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that, for all $M, L \in \mathbb{N}$, we have

$$\int_{-L}^{L} |f(t) - f_{n,L}(t)|^p \,\mathrm{d}t \leq \frac{1}{2^M} \tag{11}$$

for all $n \geq \xi(M,L)$. The set of all locally computable signals in $\mathcal{B}_{\pi}^{p}$ is denoted by $\mathcal{LCB}_{\pi}^{p}$. For $f \in \mathcal{B}_{\pi,0}^{\infty}$, we replace the condition (11) by

$$\max_{t\in[-L,L]} |f(t) - f_{n,L}(t)| \leq \frac{1}{2^M},$$

and denote the set of all locally computable signals in $\mathcal{B}_{\pi,0}^{\infty}$ by $\mathcal{LCB}_{\pi,0}^{\infty}$.

We have $\mathcal{CB}_{\pi}^{p} \subsetneq \mathcal{LCB}_{\pi}^{p}$, $p \in (1,\infty) \cap \mathbb{R}_c$, and $\mathcal{CB}_{\pi,0}^{\infty} \subseteq \mathcal{LCB}_{\pi,0}^{\infty}$, i.e, local computability in $\mathcal{B}_{\pi,0}^{\infty}$ (Definition 13) is a weaker requirement than computability in $\mathcal{B}_{\pi,0}^{\infty}$ (Definition 10). In Definition 13 we allow general computable continuous functions for the approximation, in particular, we do not require that these functions are in $\mathcal{B}_{\pi}^{p}$ or $\mathcal{B}_{\pi,0}^{\infty}$. Further, we do not demand an effective control of the approximation behavior globally, but only locally on compact intervals.

Theorem 6. *Let $p \in (1,\infty) \cap \mathbb{R}_c$. Then we have $\mathcal{CB}_{\pi}^{p} \subsetneq \mathcal{LCB}_{\pi}^{p}$. Further, we have $\mathcal{CB}_{\pi,0}^{\infty} \subseteq \mathcal{LCB}_{\pi,0}^{\infty}$.*

Remark 7. Theorem 6 shows that the requirements of Definition 13 are too weak in order to obtain a characterization of the space $\mathcal{CB}_{\pi}^{p}$. Although the requirements of Definition 13 allow us to effectively control the local approximation behavior, we have no control of $\int_{|t|>L} |f(t)|^p \,\mathrm{d}t$ that is effective in L.

Proof of Theorem 6. Let $p \in (1,\infty) \cap \mathbb{R}_c$. Further, let $\mathcal{A} \subsetneq \mathbb{N}$ be a recursively enumerable non-recursive set, and $\phi_{\mathcal{A}}\colon \mathbb{N} \to \mathcal{A}$ a recursive enumeration of $\mathcal{A}$, where $\phi_{\mathcal{A}}$ is a bijection. For $n \in \mathbb{N}$, We consider the functions

$$f_n(t) = \sum_{l=1}^{n} \frac{1}{2^{\phi_{\mathcal{A}}(l)/p}} \frac{\sin(\pi(t-l))}{\pi(t-l)}.$$

$\{f_n\}_{n\in\mathbb{N}}$ is a computable sequence of computable continuous functions. Further, let

$$f(t) = \sum_{l=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(l)/p}} \frac{\sin(\pi(t-l))}{\pi(t-l)}.$$

Since

$$\|f - f_n\|_{\mathcal{B}_{\pi}^{p}}^{p} \leq C_2(p) \sum_{k=-\infty}^{\infty} |f(k) - f_n(k)|^p = C_2(p) \left(\sum_{k=n+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(k)}} \right)^{\frac{1}{p}},$$

where we used the Plancherel–Pólya inequality [43, p. 152] in the first inequality, we see that

$$\lim_{n\to\infty} \|f - f_n\|_{\mathcal{B}_{\pi}^{p}} = 0,$$

i.e., $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\mathcal{B}_{\pi}^{p}$-norm. Let $L \in \mathbb{N}$ be arbitrary. Then, for $t \in [-L, L]$ and $n > L$, we have, for q satisfying $1/p + 1/q = 1$ that

$$\begin{aligned} |f(t) - f_n(t)| &= \left| \sum_{l=n+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(l)/p}} \frac{\sin(\pi(t-l))}{\pi(t-l)} \right| \\ &\leq \left(\sum_{l=n+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(l)}} \right)^{\frac{1}{p}} \left(\sum_{l=n+1}^{\infty} \frac{1}{\pi^q |t-l|^q} \right)^{\frac{1}{q}}. \end{aligned}$$

Since

$$\sum_{l=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(l)}} < 1$$

and

$$\begin{aligned} \left(\sum_{l=n+1}^{\infty} \frac{1}{\pi^q |t-l|^q} \right)^{\frac{1}{q}} &\leq \left(\sum_{l=n+1}^{\infty} \frac{1}{\pi^q (l-L)^q} \right)^{\frac{1}{q}} \\ &= \frac{1}{\pi} \left(\sum_{l=n+1-L}^{\infty} \frac{1}{l^q} \right)^{\frac{1}{q}} < \frac{1}{\pi} \left(\sum_{l=n+1-L}^{\infty} \int_{l-1}^{l} \frac{1}{\tau^q} \,\mathrm{d}\tau \right)^{\frac{1}{q}} \\ &= \frac{1}{\pi} \left(\int_{n-L}^{\infty} \frac{1}{\tau^q} \,\mathrm{d}\tau \right)^{\frac{1}{q}} = \frac{1}{\pi (q-1)^{\frac{1}{q}} (n-L)^{\frac{1}{p}}}, \end{aligned}$$

it follows that

$$|f(t) - f_n(t)| \leq \frac{1}{\pi (q-1)^{\frac{1}{q}} (n-L)^{\frac{1}{p}}}.$$

for all $t \in [-L, L]$ and $n > L$. Hence, we have

$$\begin{aligned} \int_{-L}^{L} |f(t) - f_n(t)| \,\mathrm{d}t &\leq 2L \max_{t\in[-L,L]} |f(t) - f_n(t)| \\ &\leq \frac{2L}{\pi (q-1)^{\frac{1}{q}} (n-L)^{\frac{1}{p}}} \end{aligned}$$

for all $n > L$, which shows that $\{f_n\}_{n\in\mathbb{N}}$ converges effectively to f on $[-L, L]$ in the L^p-norm. Hence, we see that $f \in \mathcal{LCB}_{\pi}^{p}$.

Since

$$\sum_{l=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(l)}} \notin \mathbb{R}_c$$

according to [28, p. 20, Corollary 2b], we have

$$\|f|_{\mathbb{Z}}\|_{\ell^p}^{p} = \sum_{l=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(l)}} \notin \mathbb{R}_c.$$

Further, for $k \in \mathbb{Z}$, we have

$$f(k) = \begin{cases} 2^{-\phi_{\mathcal{A}}(k)/p}, & k \geq 1, \\ 0 & k \leq 0, \end{cases}$$

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which implies that $f|_{\mathbb{Z}}$ is a computable sequence of computable numbers. Finally, Theorem 5 in [35] implies that $f \notin \mathcal{CB}_\pi^p$.

The statement $\mathcal{CB}_{\pi,0}^\infty \subset \mathcal{LCB}_{\pi,0}^\infty$ is obvious since the requirements in the definition of $\mathcal{LCB}_{\pi,0}^\infty$ are weaker than in the definition of $\mathcal{CB}_{\pi,0}^\infty$. □

XII. OVERSAMPLING

As discussed in Section VII, Definition 11 of the spaces $\mathcal{CCB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and $\mathcal{CCB}_{\pi,0}^\infty$ gives us much freedom in the choice of the approximation process. This allows us to connect these spaces with the oversampling behavior of signals in $\mathcal{B}_\pi^p$. In the case of oversampling, we can use more general kernels for the approximation, which in turn enables us to algorithmically control the convergence speed.

We will use Theorem 1 to study the behavior of signals in $\mathcal{CB}_\pi^p$ when oversampling is employed, and see that the space $\mathcal{CCB}_\pi^p$ plays an important role in this analysis. This is a further example that illustrates the usefulness of different definitions and characterizations of computable bandlimited signals.

Theorem 7. *Let $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if there exists an $\hat{a} > 1$, $\hat{a} \in \mathbb{R}_c$, such that $f \in \mathcal{CB}_{\hat{a}\pi}^p$.*

Proof. Let $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$. "⇐": Let $\hat{a} > 1$, $\hat{a} \in \mathbb{R}_c$ be such that $f \in \mathcal{CB}_{\hat{a}\pi}^p$. According to Theorem 1, we have $f \in \mathcal{CCB}_{\hat{a}\pi}^p$. Hence, due to Definition 11, f is a computable continuous function, and there exists a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\left| \|f\|_{\mathcal{B}_{\hat{a}\pi}^p}^p - \int_{-L}^{L} |f(t)|^p \, \mathrm{d}t \right| \le \frac{1}{2^M} \tag{12}$$

for all $L \ge \xi(M)$. Since $\|f\|_{\mathcal{B}_{\hat{a}\pi}^p} = \|f\|_{\mathcal{B}_\pi^p}$, (12) is also true if the $\mathcal{B}_{\hat{a}\pi}^p$-norm is replaced with the $\mathcal{B}_\pi^p$-norm. This in turn implies that $f \in \mathcal{CCB}_\pi^p$. Using Theorem 1 again, we see that $f \in \mathcal{CB}_\pi^p$. "⇒": Let $f \in \mathcal{CB}_\pi^p$. According to Theorem 1, we have $f \in \mathcal{CCB}_\pi^p$. Using the same arguments as in the first part of the proof, we see that $f \in \mathcal{CCB}_{a\pi}^p$ for all $a > 1$, $a \in \mathbb{R}_c$. Using Theorem 1 again, it follows that $f \in \mathcal{CB}_{a\pi}^p$. □

Remark 8. The proof of Theorem 7 shows for $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$, that if $f \in \mathcal{CB}_{\hat{a}\pi}^p$ for some $\hat{a} > 1$, $\hat{a} \in \mathbb{R}_c$, then we have $f \in \mathcal{CB}_{a\pi}^p$ for all $a \ge 1$, $a \in \mathbb{R}_c$.

In the proof of Theorem 7 we also have implicitly proved the following result, which is the statement of Theorem 7 for $\mathcal{CCB}_\pi^p$.

Theorem 8. *Let $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$. Then we have $f \in \mathcal{CCB}_\pi^p$ if and only if there exists an $\hat{a} > 1$, $\hat{a} \in \mathbb{R}_c$, such that $f \in \mathcal{CCB}_{\hat{a}\pi}^p$.*

For $\mathcal{CCB}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$, we have no difference between oversampling and non-oversampling, which is expected since $\hat{a} > 1$ plays no role in the definition.

For the cases $p = 1$ and $p = \infty$, we have the next two observations, which follow directly from the definitions of $\mathcal{CCB}_\sigma^1$ and $\mathcal{CCB}_{\sigma,0}^\infty$.

Observation 1. *If $f \in \mathcal{B}_\pi^1$ and $f \in \mathcal{CCB}_{a\pi}^1$ for some $a > 1$, $a \in \mathbb{R}_c$, then we have $f \in \mathcal{CCB}_\pi^1$.*

Observation 2. *If $f \in \mathcal{B}_{\pi,0}^\infty$ and $f \in \mathcal{CCB}_{a\pi,0}^\infty$ for some $a > 1$, $a \in \mathbb{R}_c$, then we have $f \in \mathcal{CCB}_{\pi,0}^\infty$.*

The goal of the rest of this section is to analyze whether, for $a > 1$, $a \in \mathbb{R}_c$, we can couple the behavior of $f \in \mathcal{CB}_{a\pi}^p$ with the behavior of $f|_{\mathbb{Z}/a}$.

Theorem 9. *Let $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$. Then for $a > 1$, $a \in \mathbb{R}_c$, the following two statements are equivalent:*

1) *We have $f \in \mathcal{CB}_{a\pi}^p$.*
2) *$f|_{\mathbb{Z}/a} = \{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers and $\|f|_{\mathbb{Z}/a}\|_{\ell^p} \in \mathbb{R}_c$.*

For the case $p = \infty$, we have a similar result.

Theorem 10. *Let $f \in \mathcal{B}_{\pi,0}^\infty$. Then for $a > 1$, $a \in \mathbb{R}_c$, the following statements are equivalent:*

1) *We have $f \in \mathcal{CB}_{a\pi,0}^\infty$.*
2) *$f|_{\mathbb{Z}/a} = \{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers and $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$.*

In Theorem 9, we have in item 2 the requirement that $\|f|_{\mathbb{Z}/a}\|_{\ell^p} \in \mathbb{R}_c$. This is a weak requirement compared to item 2 in Theorem 10, where, for $p = \infty$, we require that $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$. Note that $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$ implies $\|f|_{\mathbb{Z}/a}\|_{\ell^\infty} \in \mathbb{R}_c$. However, for $x \in c_0$ with $\|x\|_{\ell^\infty} \in \mathbb{R}_c$ we do not necessarily have $x \in \mathcal{C}c_0$.

If $x \in c_0$ and $x = \{x(k)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers, we immediately see that $\|x\|_{\ell^\infty} \in \mathbb{R}_c$. Since $\lim_{|k|\to\infty} |x(k)| = 0$, there exists a natural number $\hat{k}$ such that $\max_{k\in\mathbb{Z}} |x(k)| = |x(\hat{k})|$, and clearly we have $|x(\hat{k})| \in \mathbb{R}_c$, because $\{x(k)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers. However, using the techniques from [44], it is possible to construct an $x_* \in c_0$ such that $x_* = \{x_*(k)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers, but we have $x_* \notin \mathcal{C}c_0$. We will use the same line of arguments in the proof of Theorem 11.

Proof of Theorem 9. Let $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and $a > 1$, $a \in \mathbb{R}_c$.

We start with the case $p \in (1,\infty) \cap \mathbb{R}_c$. Since $f \in \mathcal{B}_\pi^p$, we also have $f \in \mathcal{B}_{a\pi}^p$, and the statement of the theorem follows directly from [35, Theorem 5].

It remains to prove the case $p = 1$. "⇒": Let $f \in \mathcal{CB}_{a\pi}^1$. We will show that statement 2) is true. Since $f \in \mathcal{CB}_{a\pi}^1$, there exists a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of elementary computable functions in $\mathcal{B}_{a\pi}^1$ with

$$\|f - f_n\|_{\mathcal{B}_{a\pi}^1} \le \frac{1}{2^n} \tag{13}$$

for all $n \in \mathbb{N}$. As an elementary computable function in $\mathcal{B}_{a\pi}^1$, f_n has the form

$$f_n(t) = \sum_{k=-M_n}^{M_n} c_{n,k} \frac{\sin\left(a\pi(t - \frac{k}{a})\right)}{a\pi(t - \frac{k}{a})}. \tag{14}$$

Since

$$\sum_{k=-\infty}^{\infty} \left| f_n\left(\frac{k}{a}\right) \right| = \sum_{k=-M_n}^{M_n} |c_{n,k}|,$$

according to (14), we see that

$$\left\{\sum_{k=-\infty}^{\infty}\left|f_n\left(\frac{k}{a}\right)\right|\right\}_{n\in\mathbb{N}}$$

is a computable sequence of computable numbers. Further, we have

$$\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{a}\right)\right| \leq C_3(a)\|f\|_{\mathcal{B}^1_{a\pi}} < \infty, \tag{15}$$

where we used Nikol'skiĭ's inequality [37, p. 49] in the first inequality. Using the triangle inequality, Nikol'skiĭ's inequality, and (13), we obtain

$$\begin{aligned}\left|\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{a}\right)\right| - \sum_{k=-\infty}^{\infty}\left|f_n\left(\frac{k}{a}\right)\right|\right| &\leq \sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{a}\right) - f_n\left(\frac{k}{a}\right)\right| \\ &\leq C_3(a)\|f-f_n\|_{\mathcal{B}^1_{a\pi}} \leq \frac{C_3(a)}{2^n},\end{aligned}$$

which shows that the convergence is effective. It follows that

$$\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{a}\right)\right|$$

is a computable real number. Further, since $f \in \mathcal{CB}^1_{a\pi}$, $\{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers. This completes the "$\Rightarrow$" direction of the proof.

"$\Leftarrow$": Let $f \in \mathcal{B}^1_{\pi}$ and $a > 1$, $a \in \mathbb{R}_c$ be such that $\{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers and $\|f|_{\mathbb{Z}/a}\|_{\ell^1} \in \mathbb{R}_c$. Further, let $\kappa \in \mathcal{CB}^1_{a\pi}$ be defined in the frequency domain by

$$\hat{\kappa}(\omega) = \begin{cases}\frac{1}{a}, & |\omega| \leq \pi, \\ \frac{|\omega|-a\pi}{a\pi(1-a)}, & \pi < |\omega| < a\pi, \\ 0, & |\omega| \geq a\pi.\end{cases} \tag{16}$$

Then we have

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{a}\right)\kappa\left(t-\frac{k}{a}\right), \quad t\in\mathbb{R}, \tag{17}$$

and the series in (17) converges absolutely. According to our assumption, we have $\|f|_{\mathbb{Z}/a}\|_{\ell^1} \in \mathbb{R}_c$. Thus, it follows that there exists a recursive function η such that for all $M \in \mathbb{N}$ we have

$$\sum_{|k|>N}\left|f\left(\frac{k}{a}\right)\right| \leq \frac{1}{2^M} \tag{18}$$

for all $N \geq \eta(M)$. Let $M \in \mathbb{N}$ be arbitrary but fixed, and let

$$f_N(t) = \sum_{k=-N}^{N} f\left(\frac{k}{a}\right)\kappa\left(t-\frac{k}{a}\right), \quad t\in\mathbb{R}.$$

Since $\{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers, and f_N is the finite sum of functions, each of which can be effectively approximated by elementary computable functions in $\mathcal{B}^1_{a\pi}$, it follows that $f_N \in \mathcal{CB}^1_{a\pi}$, and consequently that $\{f_N\}_{N\in\mathbb{N}}$ is a computable sequence of computable functions in $\mathcal{CB}^1_{a\pi}$. Since we have

$$f(t) - f_N(t) = \sum_{|k|>N} f\left(\frac{k}{a}\right)\kappa\left(t-\frac{k}{a}\right)$$

for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$. It follows that

$$\begin{aligned}&\sum_{l=-\infty}^{\infty}\left|f\left(\frac{l}{a}\right) - f_N\left(\frac{l}{a}\right)\right| \\ &\leq \sum_{l=-\infty}^{\infty}\sum_{|k|>N}\left|f\left(\frac{k}{a}\right)\kappa\left(\frac{l}{a}-\frac{k}{a}\right)\right| \\ &= \sum_{l=-\infty}^{\infty}\left|\kappa\left(\frac{l}{a}\right)\right| \cdot \sum_{|k|>N}\left|f\left(\frac{k}{a}\right)\right|,\end{aligned}$$

and, using Nikol'skiĭ's inequality [37, p. 49], we obtain

$$\sum_{l=-\infty}^{\infty}\left|f\left(\frac{l}{a}\right) - f_N\left(\frac{l}{a}\right)\right| \leq a(1+\pi)\|\kappa\|_1 \sum_{|k|>N}\left|f\left(\frac{k}{a}\right)\right|. \tag{19}$$

Because of (18), the right-hand side of (19) converges effectively to zero. According to [45, p. 182, Theorem 17], we have

$$\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t \leq C_4(a)\sum_{l=-\infty}^{\infty}\left|f\left(\frac{l}{a}\right)\right|,$$

and it follows that

$$\int_{-\infty}^{\infty}|f(t)-f_N(t)|\,\mathrm{d}t \leq C_4(a)a(1+\pi)\|\kappa\|_1\sum_{|k|>N}\left|f\left(\frac{k}{a}\right)\right|.$$

Hence, f is the effective limit in the $\mathcal{B}^1_{a\pi}$-norm of $\{f_N\}_{N\in\mathbb{N}}$, which is a computable sequence of computable functions in $\mathcal{CB}^1_{a\pi}$. This implies that $f \in \mathcal{CB}^1_{a\pi}$. □

Proof of Theorem 10. Let $f \in \mathcal{B}^{\infty}_{\pi,0}$ and $a > 1$, $a \in \mathbb{R}_c$. "$\Rightarrow$": Let $f \in \mathcal{CB}^{\infty}_{a\pi,0}$. Then we have $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$. "$\Leftarrow$": Let $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$. We use the same approach as in the proof of Theorem 9. Let κ be as defined in (16) and

$$f_N(t) = \sum_{k=-N}^{N} f\left(\frac{k}{a}\right)\kappa\left(t-\frac{k}{a}\right), \quad t\in\mathbb{R}.$$

Since $\{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers, and f_N is the finite sum of functions, each of which can be effectively approximated by elementary computable functions in $\mathcal{CB}^{\infty}_{a\pi}$, it follows that $f_N \in \mathcal{CB}^{\infty}_{a\pi}$. We have

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{a}\right)\kappa\left(t-\frac{k}{a}\right), \quad t\in\mathbb{R},$$

and consequently

$$\begin{aligned}|f(t)-f_N(t)| &\leq \sum_{|k|>N}\left|f\left(\frac{k}{a}\right)\kappa\left(t-\frac{k}{a}\right)\right| \\ &\leq \max_{|k|>N}\left|f\left(\frac{k}{a}\right)\right|\sum_{k=-\infty}^{\infty}\left|\kappa\left(t-\frac{k}{a}\right)\right| \\ &\leq \max_{|k|>N}\left|f\left(\frac{k}{a}\right)\right| a(1+\pi)\|\kappa\|_1,\end{aligned}$$

where we used Nikol'skiĭ's inequality [37, p. 49] in the last inequality. Since the right-hand side does not depend on t, we obtain

$$\|f - f_N\|_\infty \leq \max_{|k|>N} \left| f\left(\frac{k}{a}\right)\right| a(1+\pi)\|\kappa\|_1.$$

Since $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$, there exists a recursive function η such that for all $M \in \mathbb{N}$ we have

$$\max_{|k|>N} \left| f\left(\frac{k}{a}\right)\right| \leq \frac{1}{2^M}$$

for all $N \geq \eta(M)$. Hence, we see that $\|f - f_N\|_\infty$ converges effectively to zero. This shows that $f \in \mathcal{CB}^\infty_{a\pi,0}$. □

We will show next that item 2 in Theorem 10 cannot be weakened. Here we indeed need the condition $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$, the weaker condition $\|f|_{\mathbb{Z}/a}\|_{\ell^\infty} \in \mathbb{R}_c$ is not sufficient.

Theorem 11. *There exists a signal $f \in \mathcal{B}^\infty_{\pi,0}$ such that, for some $a > 1$, $a \in \mathbb{R}_c$, we have that $f|_{\mathbb{Z}/a} = \{f(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers and $\|f|_{\mathbb{Z}/a}\|_{\ell^\infty} \in \mathbb{R}_c$, but there exists no $a > 1$, $a \in \mathbb{R}_c$, such that $f \in \mathcal{CB}^\infty_{a\pi,0}$.*

Proof. We use the same function f_3 as in proof of Theorem 5. For this function we have $f_3 \in \mathcal{B}^\infty_{\pi,0}$, $\|f_3\|_\infty \in \mathbb{R}_c$, and f_3 is a computable continuous function on $\mathbb{R}$. Thus, since $\{k/a\}_{k\in\mathbb{Z}}$, $a > 1$, $a \in \mathbb{R}_c$ is a computable sequence of computable numbers and f_3 is a computable continuous function, it follows that $\{f_3(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers. Further, since

$$\lim_{|k|\to\infty} \left| f_3\left(\frac{k}{a}\right)\right| = 0,$$

there exists a natural number $\hat{k}$ such that

$$\max_{k\in\mathbb{Z}} \left| f_3\left(\frac{k}{a}\right)\right| = \left| f_3\left(\frac{\hat{k}}{a}\right)\right|,$$

and clearly we have

$$\left| f_3\left(\frac{\hat{k}}{a}\right)\right| \in \mathbb{R}_c,$$

because $\{f_3(k/a)\}_{k\in\mathbb{Z}}$ is a computable sequence of computable numbers. Hence, wee see that $\|f|_{\mathbb{Z}/a}\|_{\ell^\infty} \in \mathbb{R}_c$.

The rest of the proof is done indirectly. We assume that there exists an $a > 1$, $a \in \mathbb{R}_c$, such that $f_3 \in \mathcal{B}^\infty_{a\pi,0}$, and show that this assumption leads to a contradiction. Since $f_3 \in \mathcal{B}^\infty_{a\pi,0}$, there exists a recursive function $\eta\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$, we have $|f_3(t)| \leq 2^{-M}$ for all $|t| \geq \eta(M)$. This follows from a straight forward adaption of [44, Theorem 6] to signals with arbitrary bandwidth. However, in [44, Corollary 2] it was proved for f_3 that such a recursive function η cannot exist. □

The proof of Theorem 10 immediately leads to the following theorem.

Theorem 12. *Let $f \in \mathcal{CCB}^\infty_{\pi,0}$. Then we have $f \in \mathcal{CB}^\infty_{a\pi,0}$ for all $a > 1$, $a \in \mathbb{R}_c$.*

Proof. Let $f \in \mathcal{CCB}^\infty_{\pi,0}$ and $a > 1$, $a \in \mathbb{R}_c$, be arbitrary. Then we have $f|_{\mathbb{Z}/a} \in \mathcal{C}c_0$. The rest of the proof is done exactly along the same steps as in the "⇐" direction of the proof of Theorem 10. □

Combining several of our previous results, we obtain the following corollary about $\mathcal{CB}^\infty_{\pi,0}$ and oversampling.

Corollary 2. *Let $f \in \mathcal{B}^\infty_{\pi,0}$. If $f \in \mathcal{CB}^\infty_{\hat{a}\pi,0}$ for some $\hat{a} > 1$, $\hat{a} \in \mathbb{R}_c$, then we have $f \in \mathcal{CB}^\infty_{a\pi,0}$ for all $a > 1$, $a \in \mathbb{R}_c$.*

Corollary 2 is essentially the statement of Remark 8 for $p = \infty$. Note that in Corollary 2 we can only conclude that $f \in \mathcal{CB}^\infty_{a\pi,0}$ for all a that are strictly larger than one. This is in contrast to Remark 8.

Proof of Corollary 2. Let $f \in \mathcal{B}^\infty_{\pi,0}$ and $f \in \mathcal{CB}^\infty_{\hat{a}\pi,0}$ for some $\hat{a} > 1$, $\hat{a} \in \mathbb{R}_c$. Then we have $f \in \mathcal{CCB}^\infty_{\hat{a}\pi,0}$, according to Theorem 2. Further, Observation 2 implies that $f \in \mathcal{CCB}^\infty_{\pi,0}$. Application of Theorem 12 completes the proof. □

XIII. Conclusion and Open Problems

Bandlimited signals and systems are important in signal processing and system theory, because they provide a foundation for the transition between discrete-time and continuous-time signals and systems, and hence are essential for today's digital world. For the practical conversion of the discrete-time signal into the continuous-time signal it is necessary that the involved approximations and reconstructions can be performed effectively with an algorithmic control of the error.

Bandlimited signals and the impulse responses of bandlimited systems have the feature to be unlimited in time. In the present paper we developed a theory for the effective, i.e., algorithmic characterization of the time concentration behavior of bandlimited signals and systems. For the range $p \in (1,\infty) \cap \mathbb{R}_c$, which covers the practically relevant signals with finite energy, we could obtain a complete characterization. For $p = 1$ and $p = \infty$, however, it is no longer that simple. As discussed in Section II, both cases are important in system theory. Hence, we believe the following list of open problems is relevant.

Q1) Does there exist a signal $f \in \mathcal{B}^1_\pi$ such that $f \in \mathcal{CB}^1_{a\pi}$ for some $a > 1$, $a \in \mathbb{R}_c$, and $f \notin \mathcal{CB}^1_\pi$?
Q2) Does there exist a signal $f \in \mathcal{B}^\infty_{\pi,0}$ such that $f \in \mathcal{CB}^\infty_{a\pi,0}$ for some $a > 1$, $a \in \mathbb{R}_c$, and $f \notin \mathcal{CB}^\infty_{\pi,0}$?
Q3) Does there exist a signal $f \in \mathcal{B}^1_\pi$ such that $f \in \mathcal{CCB}^1_\pi$ and $f \notin \mathcal{CB}^1_\pi$?

Although the answers to these questions are open, we can establish a connection between question Q1) and question Q3): If the answer to question Q1) is positive, then the answer to question Q3) is positive.

Corollary 2 shows that question Q2) is equivalent to the question: Does there exist a signal $f \in \mathcal{B}^\infty_{\pi,0}$ such that $f \in \mathcal{CB}^\infty_{a\pi,0}$ for all $a > 1$, $a \in \mathbb{R}_c$, and $f \notin \mathcal{CB}^\infty_{\pi,0}$? There are properties of signals in $\mathcal{B}^\infty_{\pi,0}$ that suggest that the answer to this question, and hence to question Q2), is "yes".

Questions of computational complexity are not treated in the present paper. For example, different representations of computable bandlimited signals could lead to different complexities for their computation. In [46] it has been shown that even simple analog systems can lead to a complexity blowup, where, for a polynomial complexity input signal, the output signal has a high complexity. It would be interesting to study similar questions for the different representations that we have seen in this paper.

Appendix

Proof of Remark 1. We start with the case $p \in [1,\infty) \cap \mathbb{R}_c$, and show that if $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty)\cap\mathbb{R}_c$, then we also have that $f\colon \mathbb{R} \to \mathbb{R}$ is a computable continuous function according to Definition 7.

Let $p \in [1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{CB}_\pi^p$ be arbitrary. Then there exists a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of elementary computable functions in $\mathcal{B}_\pi^p$ such that $\|f - f_n\|_{\mathcal{B}_\pi^p} \le 2^{-n}$ for all $n \in \mathbb{N}$. Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary computable sequence of computable numbers. Since

$$f(t) = \int_{-\infty}^{\infty} f(\tau) \frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\, \mathrm{d}\tau, \quad t \in \mathbb{R},$$

for all $f \in \mathcal{B}_\pi^p$, it follows that

$$|f(t) - f_n(t)| \le \left(\int_{-\infty}^{\infty} |f(\tau) - f_n(\tau)|^p \,\mathrm{d}\tau\right)^{\frac{1}{p}} \times \times \left(\int_{-\infty}^{\infty} \left|\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\right|^q \,\mathrm{d}\tau\right)^{\frac{1}{q}}$$

for all $t \in \mathbb{R}$, where q, satisfying $1/q + 1/p = 1$, is the conjugate index of p. Further, since

$$\left(\int_{-\infty}^{\infty} \left|\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\right|^q \,\mathrm{d}\tau\right)^{\frac{1}{q}} = \left(\int_{-\infty}^{\infty} \left|\frac{\sin(\pi\tau)}{\pi\tau}\right|^q \,\mathrm{d}\tau\right)^{\frac{1}{q}},$$

we see that

$$C_5(q) = \left(\int_{-\infty}^{\infty} \left|\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\right|^q \,\mathrm{d}\tau\right)^{\frac{1}{q}}$$

is a computable constant. For $p = 1$ we have to choose $q = \infty$ and the L^q-norms in the above expressions are replaced by the L^∞-norm. For $k \in \mathbb{N}$ and $n \in \mathbb{N}$, $\zeta_{n,k} = f_n(t_k)$ defines a computable double sequence of computable numbers, because every elementary computable function is a computable continuous function [28, p. 27]. Further, according to the above calculation, we have

$$|f(t_k) - \zeta_{n,k}| \le \frac{1}{2^n} C_5(q).$$

This shows that $\{f(t_k)\}_{k\in\mathbb{N}}$ is a computable sequence of computable numbers [28, p. 20, Proposition 1]. This proves condition 1 of Definition 7.

Next, we prove condition 2. According to Bernstein's inequality [37, Theorem 6.7, p. 49], we have $\|f'\|_{\mathcal{B}_\pi^p} \le \pi\|f\|_{\mathcal{B}_\pi^p}$. Further, we have $\|f\|_{\mathcal{B}_\pi^p} \le \|f_1\|_{\mathcal{B}_\pi^p} + 1/2$. Hence, it follows that

$$\|f'\|_{\mathcal{B}_\pi^p} \le \pi\left(\|f_1\|_{\mathcal{B}_\pi^p} + \frac{1}{2}\right)$$

and consequently that

$$|f'(t)| \le C_5(q)\pi\left(\|f_1\|_{\mathcal{B}_\pi^p} + \frac{1}{2}\right)$$

for all $t \in \mathbb{R}$. The mean value theorem gives

$$\begin{aligned} |f(t_1) - f(t_2)| &= |f'(\tau)| \cdot |t_1 - t_2| \\ &\le C_5(q)\pi\left(\|f_1\|_{\mathcal{B}_\pi^p} + \frac{1}{2}\right)|t_1 - t_2|. \end{aligned}$$

Let $\hat{k} \in \mathbb{N}$ be such that

$$C_5(q)\pi\left(\|f_1\|_{\mathcal{B}_\pi^p} + \frac{1}{2}\right) < 2^{\hat{k}}.$$

Let $M \in \mathbb{N}$ be arbitrary. Then, for all t_1, t_2 with $|t_1 - t_2| < 1/2^{\hat{k}+M}$, we have $|f(t_1) - f(t_2)| \le 1/2^M$. That is, condition 2 of Definition 7, even independently of L.

For $p = \infty$, i.e., $f \in \mathcal{CB}_{\pi,0}^\infty$, we use the inequality $|f(t) - f_n(t)| \le \|f - f_n\|_{\mathcal{B}_\pi^\infty}$, which holds for all $t \in \mathbb{R}$. The rest of the proof is done analogously to the proof of the case $p \in [1,\infty) \cap \mathbb{R}_c$. □

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